Nonlocal Morphological Levelings by Partial Difference Equations over Weighted Graphs

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Abstract

In this paper, a novel approach to Mathematical Morphology operations is proposed. Morphological operators based on partial differential equations (PDEs) are extended to weighted graphs of the arbitrary topologies by considering partial difference equations. We focus on a general class of morphological filters, the levelings; and propose a novel approach of such filters. Indeed, our methodology recovers classical local PDEs-based levelings in image processing, generalizes them to nonlocal configurations and extends them to process any discrete data that can be represented by a graph. Experimental results show applications and the potential of our levelings to textured image processing, region adjacency graph based multiscale leveling and unorganized data set filtering.

1. Introduction

Mathematical Morphology (MM) operations modeled by partial differential equations (PDEs) have shown their efficiency and their flexibility to address several tasks in computer vision [1, 5]. The two fundamental MM operators are *dilation* and *erosion*. Given a symmetric unit disc $\beta := \{z \in \mathbb{R}^2 \colon |z|_2 \leq 1\}$, the multiscale flat dilation δ and erosion ϵ of an image $f^0 \colon \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are generated by the following PDEs: $\delta_t(f) = +|\nabla f|_2$ and $\epsilon_t(f) = -|\nabla f|_2$ where ∇ is the spatial gradient operator and f the transformed version of f^0. Based on such basic operations, Maragos [6] defines a PDEs-based general class of morphological filters, the levelings which include the reconstruction openings and closings. Let f^0 be an initial function and m a marker function, the PDE generating levelings is $\partial f / \partial t = \mathrm{sgn}(f^0 - f)|\nabla f|_2$ where sgn is the sign function and $f = m$ at $t = 0$ is the initial condition. This class of morphological filters have found many applications for image enhancement, simplification or segmentation. Such PDEs-based methods have the advantages of better mathematical modeling, more connections with physics, better geometry approximation and subpixel accuracy. Nevertheless, these methods have several drawbacks. First, their numerical algorithms require a careful choice of spatial discretization which is difficult for high dimensional data or irregular domains. Second, these approaches only consider local derivatives while nonlocal schemes have recently shown their effectiveness for image processing [3, 2, 4]. Finally, MM is a well known method for binary and grayscale image but there exist no general extension for multivariate high dimensional data processing.

Paper contribution. From the works investigated in [3], we propose a novel approach to MM operations by extending PDEs-based methods to nonlocal discrete schemes over weighted graphs. To this aim, we introduce nonlocal discrete derivatives and partial difference equations over graphs. Thus, our graph-based MM framework recovers local PDEs-based approaches, generalizes them for nonlocal configurations and extends them to process any discrete data that can be represented by a weighted graph without any spatial discretization. In this paper, we focus on the case of levelings defined within our framework and show the applications and the benefits of such novel morphological filters for image and unorganized discrete data processing.

2. Nonlocal dilation and erosion

This Section recalls notations on graphs; introduces our nonlocal morphological graph-based framework and defines a family of dilation and erosion processes based on nonlocal gradients over graphs.

Weighted graphs. We consider that any discrete domain can be represented by a weighted graph $G=(V,E,w)$ composed of a set of *vertices* V, a set of *edges* $E \subseteq V \times V$, and a *weight function* $w\colon V \to \mathbb{R}^+$. An edge of E which connects two adjacent vertices u and v is noted uv. In this work, G is considered as simple, connected and undirected. This implies that w is symmetric: $w_{uv}=w_{vu}$ if $uv \in E$. We assume that any function $f\colon V \to \mathbb{R}$ with $f \in \mathcal{H}(V)$ assigns a real value $f(u)$ for each $u \in V$, where $\mathcal{H}(V)$ is the Hilbert space of real valued functions defined on V.

Nonlocal discrete gradient operators. From the operators defined in [3], the *directional derivative* of a function $f\colon V \to \mathbb{R}$ defined at vertex u along an edge $uv \in E$ is $\partial_v f(u) = w_{uv}^{1/2}\big(f(v)-f(u)\big)$. With this definition, we obtain two other derivatives based on $\min$ and $\max$ operators,

$$\begin{aligned} &\partial_v^+ f(u) = w_{uv}^{1/2}\max\Big(0, f(v)-f(u)\Big) \text{ and} \\ &\partial_v^- f(u) = w_{uv}^{1/2}\min\Big(0, f(v)-f(u)\Big) . \end{aligned} \quad (1)$$

The *weighted gradient operators* of f at $u \in V$, for each derivatives from definitions (1), are defined by

$$\nabla_w^{\pm} f(u) := \Big(\partial_v^{\pm} f(u) : u \sim v\Big)^T \quad \forall uv \in E , \quad (2)$$

where $u \sim v$ means that vertex v is adjacent to u, and $\nabla_w^+ f$ (resp. $\nabla_w^- f$) is defined with $\partial_v^+ f$ (resp. $\partial_v^- f$). To compute these *gradient norms*, the $\mathcal{L}_p$-norm is used. From definition (2), for $u \in V$ and $0 < p < +\infty$, it leads to

$$\Big|\nabla_w^{\pm} f(u)\Big|_p = \Bigg[\sum_{u \sim v}\Big|\partial_v^{\pm} f(u)\Big|^p\Bigg]^{1/p} . \quad (3)$$

Dilation and erosion processes. We define a discrete analogue of the continuous PDEs-based dilation and erosion formulations of a function $f \in \mathcal{H}(V)$. To this aim, we define the notion of graph boundary. Let $A \subset V$ be a set of connected vertices. For a given vertex $u \in V$, we denote by $\partial^+ A = \big\{u \in A^c\colon \exists\, v \in A,\, v \sim u\big\}$ and $\partial^- A = \big\{u \in A\colon \exists\, v \in A^c,\, v \sim u\big\}$, respectively the *outer* and the *inner* boundaries of A. A^c is the complement of A. Then, dilation over A is a growth process that adds vertices from $\partial^+ A$ to A. By duality, erosion is a contraction process that removes vertices from $\partial^- A$. This property can be demonstrated by using levels sets decomposition of f[8]. A variational definition of dilation (resp. erosion) process can be interpreted as maximizing (resp. minimizing) a surface gain proportionally to $+\big|\nabla_w^+ f\big|_p$ (resp. $-\big|\nabla_w^- f\big|_p$). Over a graph G, these processes lead to a family of p-dilation and p-erosion parameterized by p and w. For $0 < p < +\infty$ they are defined by,

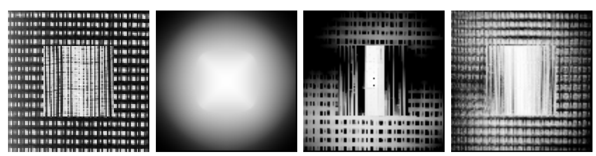
$$\delta_{p,t} := \partial f/\partial t = +\Big|\nabla_w^+ f\Big|_p \quad \text{and} \quad \epsilon_{p,t} := \partial f/\partial t = -\Big|\nabla_w^- f\Big|_p , \quad (4)$$

where t corresponds to an artificial time parameter. To solve equations in (4), on the contrary to the PDEs case, no spatial discretization is needed thanks to derivatives directly expressed in a discrete form. With the conventional notation $f^n \approx f(u, n\Delta t)$, one obtains the following iterative algorithm for p-dilation and p-erosion of an initial function $f^0 \in \mathcal{H}(V)$. For all $u \in V$ and $0 < p < +\infty$,

$$\begin{cases} f^{n+1}(u) = f^n(u) \pm \Delta t\Big[\sum\limits_{u \sim v} w_{uv}^{1/2}\Big|\mathrm{M}^{\pm}\Big(0, f^n(v)-f^n(u)\Big)\Big|_p\Big]^{1/p} \\ f^{(0)}(u) = f^0(u) \end{cases} \quad (5)$$

where n is the iteration step. The plus (resp. minus) sign corresponds to p-dilation (resp. p-erosion) and one uses the $\mathrm{M}^+ = \max$ (resp. $\mathrm{M}^- = \min$) operator.

Local and nonlocal configurations. One can note that Algorithm (5) enables local and nonlocal configurations. Indeed, the choice of the graph topology models local or nonlocal interactions between data. These interactions are directly integrated into edge weights by the associated weight function w [3]. Then, adaptive operations are naturally expressed by both weight function and graph topology. Fig. 1 illustrates this behavior. It



(a) original (b) unweighted (c) weighted (d) nonlocal + patch

Figure 1. Adaptive dilation of a pulse centered on the image

shows a p-dilation (p=2) of a pulse located on the center of a scalar grayscale image $f^0\colon V \subset \mathbb{R}^2 \to \mathbb{R}$. Results are obtained with Algorithm (5) where the edge weights are computed from original image (Fig. 1(a)). In Figs. 1(b) and 1(c), the considered graph is a 4-adjacency grid graph where each vertex corresponds to an image pixel. In Fig. 1(b), the graph is unweighted, w_{uv}=1; whereas in Fig. 1(c), $w_{uv} = \exp\big(-\big|f(u)-f(v)\big|^2/\sigma^2\big)$. Fig. 1(d) shows a dilation obtained with a nonlocal k-Nearest Neighbors (k-NN) graph based on patch distance. In this case, each vertex $u \in V$ is defined by a feature vector $F(f^0,u) = \big[f^0(v)\colon v \in B_{u,s}\big]^T$, where $B_{u,s}$ is a bounding box of size s centered on u. It defines a patch of size $(2s+1) \times (2s+1)$. Then, the patch distance between two vertices u and v is computed by the Euclidean distance weighted by a Gaussian kernel. In Fig. 1(d), k is equal to 10, patch size is 7×7 and search window size is 21×21 to select the k nearest neighbors. Results show the benefits of weights to preserve image features, while a nonlocal patch-based approach better detect fine and repetitive image structures.

3. Nonlocal levelings for discrete data

Levelings are a general class of morphological filters. A particular case of such filters is reconstruction openings and closings [6]. In this Section, by using previous definitions, we define the discrete analogue of PDEs-based morphological levelings over graphs of the arbitrary topologies. Consider an initial function $f^0 \in \mathcal{H}(V)$ and a marker function $m \in \mathcal{H}(V)$ from which a leveling can be produced. Then, the general leveling of f with respect to the marker m is computed by the following process. For $u \in V$ and sgn the sign function,

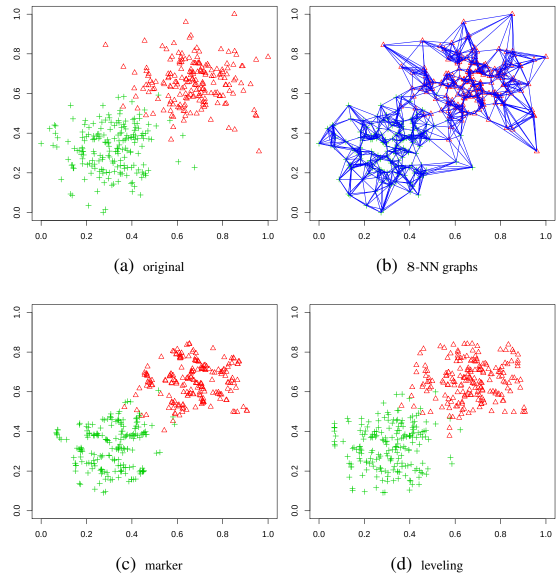
$$\begin{cases} \frac{\partial f(u)}{\partial t} = \operatorname{sgn}\big(f^0(u)-f(u)\big)\left|\nabla_w^{\pm} f(u)\right|_p \\ f(u) = m(u) \quad \text{at} \quad t=0\ . \end{cases} \tag{6}$$

When $\operatorname{sgn}\big(f^0(u)-f(u)\big)$ is equal to $+1$ (resp. -1) then Eq. (6) corresponds to a p-dilation (resp. p-erosion). One uses $\nabla_w^{+} f$ (resp. $\nabla_w^{-} f$) with the corresponding Algorithm (5) to solve the leveling process. One can remark that for the case where $p = 2$, $w = 1$ with a 4-adjacency grid graph, from Eq. (6) and Algorithm (5), one obtains exactly the leveling numerical scheme proposed by [6] in the context of image processing.

Our methodology provides a novel approach of morphological levelings and has several advantages. No spatial discretization is needed contrary to the continuous case. The choice of the graph topology provides a natural adaptive scheme. The same scheme works on graphs of the arbitrary topologies i.e. we can process in the same way any discrete data that can be represented by a graph. In the latter, we show several applications of the proposed leveling for image and unorganized data processing. The main purpose of the following experiments is not to solve a particular image processing or data analysis problems but only to illustrate the potential of our method to address several problems.

For simplicity, we restrict ourselves to use Eq. (6) for the case of $p = 2$. The proposed levelings can be used to treat any function $f^0\colon V \subset \mathbb{R}^n \rightarrow \mathbb{R}^q$. The leveling of vector-valued functions is performed as follows. For all $u \in V$, we define $f^0(u)\colon \left[f_i^0, \ldots, f_q^0\right]^T$, where $f_i^0(u)\colon V \rightarrow \mathbb{R}$ is the ith component of $f^0(u)$. Then, the levelings of vector-valued functions consist in q independent schemes where the weight function acts as a coupling term.

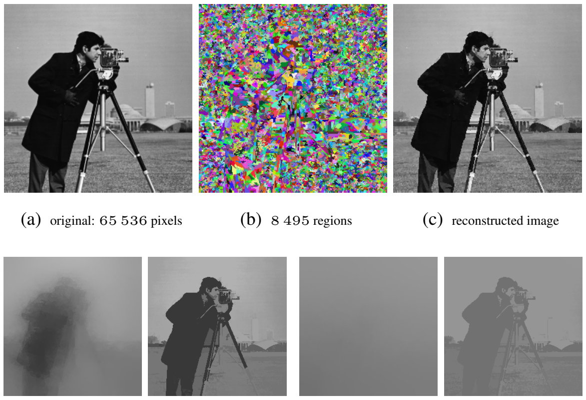
Unorganized data levelings. One of the advantages of the proposed method is: any discrete data that can be represented by a graph can be addressed. In particular, it permits to consider multivariate unorganized data set and open a novel field of application of morphological filters. Fig. 2 shows an example of unorganized data leveling. Original data consists in two noisy gaussians (Fig. 2(a)) over which an 8-NN graph is computed (Fig. 2(b)). Marker function (Fig. 2(c)) is obtained by a discrete linear diffusion filtering as in [3]. Fig. 2(d) shows the leveling result with respect to the marker. One can observe the filtering effect of the leveling: main data structures are recovered during the reconstruction and, extremum and outliers data have been filtered.



(a) original (b) 8-NN graphs (c) marker (d) leveling

Figure 2. Unorganized data set leveling

Fast multiscale RAG levelings. Our morphological framework works on graphs of the arbitrary topologies. It permits to consider other image representations and to use more high level structures, such as image regions, rather than image pixels. For instance, one can use the image-based Region Adjacency Graph (RAG) associated with a fine partition, where vertices represent regions and edges link adjacent regions. This graph representation provides a fast scheme (due to the reduced number of data to consider) to perform several tasks such as filtering [3] or image segmentation [7]. Maragos [6] shows that a multiscale leveling can be produced from various markers. Fig. 3 shows an example of analogue multiscale levelings but performed on a RAG representation. Fig. 3(c) shows a reconstructed image where each pixel value in the fine partition (Fig. 3(b)) is replaced by its surrounding region mean color value from the original image Fig. 3(a). One can note the significant data reduction of the simplified version as compared to the original one (87% in terms of vertices). The RAG-based multiscale leveling is performed with the two markers of Figs. 3(d) and 3(f).



(a) original: 65 536 pixels (b) 8 495 regions (c) reconstructed image

(d) marker 1 (e) leveling 1 (f) marker 2 (g) leveling 2

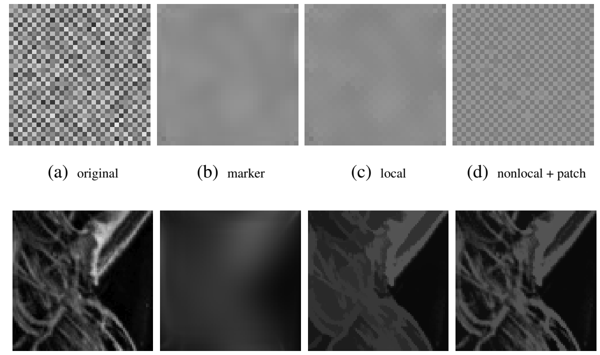
Figure 3. Fast multiscale RAG leveling

These markers are obtained by filtering the RAG using methods described in [3]. Figs. 3(e) and 3(g) show the obtained levelings with respect to the corresponding marker. Results show same behaviors as in pixel-based case while drastically reducing computation complexity. This multiscale RAG-based leveling can be viewed as region merging and region filtering processes.

Nonlocal patch-based image levelings. Our morphological framework enables nonlocal configurations. It provides a novel approach to morphological levelings. Fig. 4 shows examples of local and nonlocal patch-based levelings. Local levelings (Figs. 4(c) and 4(g)) are obtained by considering an unweighted 4-adjacency grid graph. Nonlocal patch-based leveling (Figs. 4(d) and 4(h)) are obtained with a k-NN graph based on the patch distance, with a patch of size 7×7 and a 21×21 search window size to select the $k = 10$ nearest neighbors. Results are obtained with markers shown in Fig. 4(b) and 4(f). One can note that nonlocal patch-based approach better detect image frequent and fine structures during the reconstruction process as compared to the local one.

4. Conclusion

In this paper, a novel morphological framework with partial difference equations over weighted graphs of the arbitrary topologies has been proposed. This framework generalizes PDEs-based methods to discrete local and nonlocal schemes, extends them to the treatment of any discrete data that can be represented by a graph. This work has also focused on a general class of morphological filters, the levelings. Experiments have shown novel aspects of such filters, the nonlocal patch-based approach and applications on image-based RAG and unorganized data set.



(a) original (b) marker (c) local (d) nonlocal + patch

(e) original (f) marker (g) local (h) nonlocal + patch

Figure 4. Nonlocal image leveling

5. Acknowledgment

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