# PARTIAL DIFFERENCE EQUATIONS ON GRAPHS FOR MATHEMATICAL MORPHOLOGY OPERATORS OVER IMAGES AND MANIFOLDS

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## ABSTRACT

The main tools of Mathematical Morphology are a broad class of nonlinear image operators. They can be defined in terms of algebraic set operators or as Partial Differential Equations (PDEs). We propose a framework of partial difference equations on arbitrary graphs for introducing and analyzing morphological operators in local and non local configurations. The proposed framework unifies the classical local PDEs-based morphology for image processing, generalizes them for non local configurations and extends them to the processing of any discrete data living on graphs.

***Index Terms—*** Mathematical Morphology, PDEs, Partial difference, Graphs, Non local.

## 1. INTRODUCTION

Nonlinear scale-space approaches based on Mathematical Morphology operators are one of the most important tools in image processing. Dilation and erosion are the two fundamental operators. They form the basis of many other morphological processes such as opening, closing, reconstruction, levelings, etc [1]. Let $f^0\colon\mathbb{R}^2\to\mathbb{R}$ be a function representing a 2-D signal and let a structuring function $g\colon B\to\mathbb{R}^2$ representing some structuring element with a compact support $B\subseteq\mathbb{R}^2$. Dilation $\delta$ and erosion $\varepsilon$ of $f^0$ by $g$ are defined as $\delta_g(f^0) = \max\{f^0(x-x',y-y')+g(x',y')\}$ and $\varepsilon_g(f^0) = \min\{f^0(x+x',y+y')-g(x',y')\}$ with $x,y\in f^0$ and $x',y'\in B$. In this paper we consider the flat morphology case that use structuring set, i.e. for all $x',y'\in B$ the function $g(x',y')$=0. Such dilation and erosion operators are frequently implemented by algebraic set operations. For convex structuring elements, an alternative formulation in terms of Partial Differential Equations (PDEs) has also been proposed [2]. Given a disc $B=\{z\in\mathbb{R}^2 : |z|<1\}$ one considers the following evolution equation $\partial_t f=\pm|\nabla f|$ where $\nabla=(\partial x,\partial y)^T$ denotes the spatial nabla operator. Moreover, if one assumes that at time $t$=0 the evolution is initialized with $f(x,y,0)=f^0(x,y)$, solution of $f(x,y,t)$ at time $t$>0 provides dilation (with the plus sign) or erosion (with the minus sign) within a disc of radius $t$. Such PDEs produce continuous scale morphology and offer several advantages [2]. First, they offer excellent results for non-digitally scalable structuring elements whose shapes cannot be represented correctly on a discrete grid (e.g. discs or ellipses). Second, they allow sub-pixel accuracy and can be adaptive by introducing a local speed evolution term [3]. However, they have several drawbacks. First, the numerical discretization requires a careful choice of the time and spatial size. Second, this discretization is difficult for high-dimensional data or irregular domains. Finally, they only consider local interactions on the data by using local derivatives while non local schemes have recently received a lot of attention [4, 5].

We propose to extend such PDEs-based operators to a discrete and non local scheme by considering partial difference equations over weighted graphs of the arbitrary topologies. To this aim, we explicitly introduce discrete non local derivatives on graphs to transcribe such continuous operators to operators based on graphs. Our proposal has the following advantages. No discretization step is needed and any data lying on any domain (even irregular) can be processed. The same formalism directly integrates local to non local information from data. This paper is organized as follows. Section 2 introduces partial difference equations over weighted graphs. In Section 3, we propose a family of non local weighted dilation and erosion (the *p-dilation* and the *p-erosion*). Section 4 provides several experiments for the case of image and manifold processing. Finally, Section 5 concludes.

## 2. PARTIAL DIFFERENCE EQUATIONS ON WEIGHTED GRAPHS

### 2.1. Definitions and Notations

We consider any general discrete domain as a weighted graph $G=(V,E,w)$ composed of a finite set of vertices $V=\{u_1,\dots,u_n\}$ and a finite subset of edges $E\subseteq V\times V$. An *edge* of $E$ which connects the vertices $u$ to $v$ is denoted $uv$. $G$ is weighted if it is associated with a weight function

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$w\colon E\to\mathbb{R}^+$ satisfying $w_{uv}{=}0$ if $uv\notin E$, and $w_{uv}{=}w_{vu}$ otherwise. We assume that $G$ is connected, simple, undirected and weighted. We consider that a function $f\colon V\to\mathbb{R}$ is provided that assigns a real value $f(u)$ to each vertex $u\in V$.

### 2.2. A Family of Discrete Gradient

We consider the *directional derivative* of $f$ at vertex $u$ along the edge $uv\in E$ to be defined as in [5]:

$$\left.\frac{\partial f}{\partial uv}\right|_u = \partial_v f(u) = w_{uv}^{1/2}\big(f(v)-f(u)\big)\ .$$

It satisfies $\partial_v f(u){=}{-}\partial_u f(v)$, $\partial_v f(v){=}0$, and if $f(u){=}f(v)$ then $\partial_v f(u){=}0$. Using these definitions, we can define two other directional derivatives:

$$\begin{aligned}\partial_v^+ f(u)&=\max\big(0,\partial_v f(u)\big)=w_{uv}^{1/2}\max\big(0,f(v)-f(u)\big)\\ \partial_v^- f(u)&=\min\big(0,\partial_v f(u)\big)=w_{uv}^{1/2}\min\big(0,f(v)-f(u)\big)\end{aligned}\quad(1)$$

*The weighted gradient operator* of $f$, at vertex $u\in V$, is the following column vector. For all edges $(uv_i)\in E$:

$$\begin{aligned}\nabla_w f(u)&=\big(\partial_v f(u)\colon v\sim u\big)^T=\big(\partial_{v_1} f(u),\dots,\partial_{v_i} f(u)\big)^T\\ \nabla_w^\pm f(u)&=\big(\partial_v^\pm f(u)\colon v\sim u\big)^T=\big(\partial_{v_1}^\pm f(u),\dots,\partial_{v_i}^\pm f(u)\big)^T\end{aligned}\quad(2)$$

where notation $v\sim u$ means that the vertex $v$ is adjacent to the vertex $u$, and $\nabla_w^+$ (resp. $\nabla_w^-$) is defined by using the corresponding edge derivative definition in Eq. (1). The norm of these vectors represent the *local variation* of $f$ at a vertex of the graph. Several norms can be used but we focus on the $\mathcal{L}_p$-norm. Hence, the following discrete norm of the gradients in Eq. (2) can be considered. For any function $f$ defined on $V$ and for $u\in V$, when $0<p<+\infty$

$$\begin{aligned}\big|\nabla_w^\pm f(u)\big|_p&=\Big[\sum_{v\sim u}\big|\partial_v^\pm f(u)\big|^p\Big]^{1/p}\text{ then,}\\ \big|\nabla_w^+ f(u)\big|_p&=\Big[\sum_{v\sim u} w_{uv}^{p/2}\big|\max\big(0,f(v)-f(u)\big)\big|^p\Big]^{1/p}\text{ and}\\ \big|\nabla_w^- f(u)\big|_p&=\Big[\sum_{v\sim u} w_{uv}^{p/2}\big|\min\big(0,f(v)-f(u)\big)\big|^p\Big]^{1/p}\ .\end{aligned}\quad(3)$$

When $p{=}\infty$, one obtains:

$$\begin{aligned}\big|\nabla_w^\pm f(u)\big|_\infty&=\max_{v\sim u}\big(\partial_v^\pm f(u)\big)\text{ then,}\\ \big|\nabla_w^+ f(u)\big|_\infty&=\max_{v\sim u}\Big(w_{uv}^{1/2}\big|\max\big(0,f(v)-f(u)\big)\big|\Big)\text{ and}\\ \big|\nabla_w^- f(u)\big|_\infty&=\max_{v\sim u}\Big(w_{uv}^{1/2}\big|\min\big(0,f(v)-f(u)\big)\big|\Big)\ .\end{aligned}\quad(4)$$

Same definitions can be provided for the gradient $\nabla_w f$.

## 3. A FAMILY OF DILATIONS AND EROSIONS ON WEIGHTED GRAPHS

Let $A$ be a subset of $V$. We denote by $A^+$ and $A^-$ respectively the *outer* and the *inner* boundary sets of $A$ where $A^+{=}\big\{u\in A^c\colon\exists\, v{\in}A, v{\sim}u\big\}$ and $A^-{=}\big\{u{\in}A\colon\exists\, v{\in}A^c, v{\sim}u\big\}$ where $A^c$ is the complement of $A$. Dilation over $A$ is a growth process that adds vertices from $A^+$ to $A$. Erosion over $A$ is a contraction process that removes vertices from $A^-$ to $A$.

Any function $f$ can be decomposed into its levels sets $f^k{=}H(f{-}k)$ where $H$ is the Heaviside function. Then, one can find a set $A^k{\subset}V$ such that $f^k=\chi_{A^k}$ where $\chi\colon V\to\{0,1\}$ is the indicator function. As for the continuous case, a simple variational definition of a dilation applied to $f^k$ can be interpreted as maximizing a surface gain proportional to $+\big|\nabla_w f^k(u)\big|_p$, the gradient of $f^k$. On can demonstrate[1] from Eq. (3) that for $0<p<+\infty$,

$$\begin{aligned}\big|\nabla_w^+ f^k(u)\big|_p&=\Big[\sum_{\substack{v\sim u\\ v\in A}} w_{uv}^{p/2}\Big]^{1/p}\chi_{(A^+)^k}(u)\text{ and}\\ \big|\nabla_w^- f^k(u)\big|_p&=\Big[\sum_{\substack{v\sim u\\ v\in A}} w_{uv}^{p/2}\Big]^{1/p}\chi_{(A^-)^k}(u)\ .\end{aligned}$$

In a same way, we also have

$$\big|\nabla_w f^k(u)\big|_p=\big|\nabla_w^+ f^k(u)\big|_p+\big|\nabla_w^- f^k(u)\big|_p\ .$$

Therefore, $\big|\nabla_w f^k\big|_p$ can be reduced to $\big|\nabla_w^+ f^k\big|_p$ for $u{\in}(A^+)^k$ which corresponds to a dilation on $A^k$ and can be expressed by $\partial_t f^k(u)=\big|\nabla_w^+ f^k\big|_p$. By extending this to all the levels of $f$, one obtains a $p$-dilation process over a graph. Finally, with these properties, we can naturally consider the following family (parameterized by $p$ and $w$) of $p$-dilation and $p$-erosion over a weighted graph,

$$\delta_{p,t}(f):=\partial_t f=+\big|\nabla_w^+ f\big|_p\text{ and }\varepsilon_{p,t}(f):=\partial_t f=-\big|\nabla_w^+ f\big|_p$$

where $t\geq 0$ corresponds to a scale parameter. By using discretization in time, and with the usual notation $f^n(u)\approx f(u,n\Delta t)$, we obtain the following iterative algorithms for the $p$-dilation. For all $u\in V$, when $0<p<+\infty$

$$f^{n+1}(u){=}f^n(u){+}\Delta t\sum_{v\sim u} w_{uv}^{p/2}\big|\max\big(0,f^n(v){-}f^n(u)\big)\big|^p$$

and when $p{=}\infty$

$$f^{n+1}(u){=}f^n(u){+}\Delta t\max_{v\sim u}\Big(w_{uv}^{p/2}\big|\max\big(0,f^n(v){-}f^n(u)\big)\big|\Big)\ .$$

Where $n$ corresponds to the iteration step, $f^0\colon V\to\mathbb{R}$ is the initial function defined on $V$, and the initial condition is $f^{(0)}(u){=}f^0(u)$. A similar scheme is obtained for the $p$-erosion. At each iteration step, the time complexity is $\mathcal{O}(|V|^2)$, where $|.|$ stands for the cardinality of a set.

*Remarks*. In the particular case of a grayscale image, with a 4-adjacency grid graph associated to the image (one vertex per pixel) and a constant weight function ($w{=}1$), our approach corresponds exactly to the conventional Osher-Sethian

[1] Proof can be obtained by studying cases where $u{\in}A^k$ or $u{\notin}A^k$ and similarly for $v{\sim}u$.

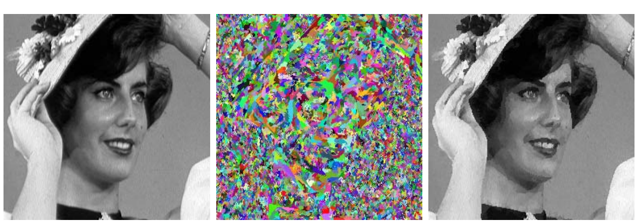
upwind discretization scheme [2] when $p{=}2$. Moreover, for the case where $p{=}\infty$ and $\Delta t{=}1$, one recovers the classical algebraic morphological formulation overs graphs.

## 4. EXPERIMENTS

The proposed family of $p$-dilation and $p$-erosion can be applied on any function defined on a discrete data set which can be represented by a weighted graph. The application of dilation or erosion on a vector valued function $f^0\colon V \subset \mathbb{R}^q \to \mathbb{R}^r$ where for $u \in V$, $f^0(u){=}\big[f_1^0(u), \ldots, f_r^0(u)\big]^T$ and $f_i^0\colon V \to \mathbb{R}$ is the $i$th component of $f^0(u)$ consists in $r$-independent iterative dilation or erosion schemes described in Section 3. In this case, the weight function of the graph acts as a coupling term between each vector components.

### 4.1. Image Processing

Let $f^0\colon V \subset \mathbb{R}^2 \to \mathbb{R}$ be a scalar image which defines a mapping from the vertices to gray levels. Fig. 1 presents an initial image, a fine partition of this image and a reconstructed image from the partition where the pixel values of each region of the partition are replaced by the mean pixel value of its region. To obtain the fine partition, many well known methods can be used. in this work an approach based on the generalized Voronoï diagram [6] is used. The amortized time complexity to obtain such partition is $\mathcal{O}(E{+}V \log V)$ with Dikjstra algorithm and Fibonacci heap structure. Then, the partition can be associated with a Region Adjacency Graph (RAG), where vertices represent regions and where edges link adjacent regions. One can note the significant data reduction of the reconstructed version as compared to the original one ($88\%$ of reduction in terms of vertices).



(a) (b) (c)

**Fig. 1**. (a) Original image of size $256{\times}256{=}65\,536$ pixels. (b) fine partition of (a) having $8\,114$ regions (i.e. $88\%$ of reduction). (c) reconstructed image from (b).

Fig. 2 illustrates the application of the proposed family of $p$-dilation and $p$-erosion on gray level image with different values of $p$, weight functions and graph topologies. In Fig. 2. For each morphological processing ($p$-dilation or $p$-erosion) a table view of the results is provided (with the same number of iterations). Each first and second rows show a local processing performed on a 4-adjacency grid graph ($\mathcal{N}_4$) associated to the initial image with constant weights $w{=}1$ (the first row corresponds to classical approaches) and non constant weights $w{=}w_e{=}w_{uv}{=}\exp\big(\big|f(u){-}f(v)\big|_2^2/\sigma^2\big)$ (second row) where $\big|f(u){-}f(v)\big|_2$ is the $\mathcal{L}_2$-norm between feature vectors. Third row presents results with non local processing on a 48-adjacency graph ($7{\times}7$ neighborhood window) with a $3{\times}3$-patch as a feature vector (denoted $\mathcal{N}_{48,9}$) and non constant weights ($w{=}w_e$). Fourth and last rows present results on the Region Adjacency Graph of the image partition with constant weights (fourth row) and non constant weights $w{=}w_e$ (last row).

These results show that by using non constant weights, the proposed $p$-dilation and $p$-erosion preserve better preserve edges as compared to classical approaches. Once one uses a non local configuration, one also better preserve fine structures and repetitive elements. In addition, our formulation works with equal ease on graphs of the arbitrary topology and, therefore, we can apply the same schemes on any graph representing the image. For instance, we can use the above mentioned RAG (Figs. 1(b) and 1(c)) and associate to each vertex the mean gray level of its corresponding region. This exhibits similar behaviors (the two last rows in Fig. 2) while reducing complexity due to the reduced number of vertices to consider.

### 4.2. Manifold Processing

To illustrate the generality of our formulation, we apply our family of $p$-dilation and $p$-erosion on an unorganized data set (the Iris data set[2]). Fig. 3 presents results of a $p$-dilation and a $p$-erosion for $p{=}1$ with a non constant weight function $w_{uv}{=}1/\big(\big|f(u){-}f(v)\big|_2^2\big)$. The graph used to represent data corresponds to a $k$-nearest neighbors graph ($k{=}30$). Iris data set contains 3 classes of samples in 4-dimensions, with 50 samples in each class. Fig. 3 shows projection of the two features (for better visualization) but it is important to note that weight function takes into account all the data features. Therefore, the proposed morphological processing of Iris data set consists in 4 independent $p$-erosion or $p$-dilation schemes. Fig. 3 shows the evolution of the processing on the data for different number $n$ of iterations. One can note that the dilation and the erosion tend to naturally group the data in different parts of the feature space and to move data to respectively the maximum and the minimum of the feature space.

## 5. CONCLUSION

In this paper, we presented a new formalism that extends PDEs-based Mathematical Morphology operators to a discrete and non local scheme by considering partial difference equations over weighted graphs of the arbitrary topologies. We introduced basic operators: dilation and erosion that can

[2]UCI Machine Learning Repository, http://www.ics.uci.edu/~mlearn/MLRepository.html

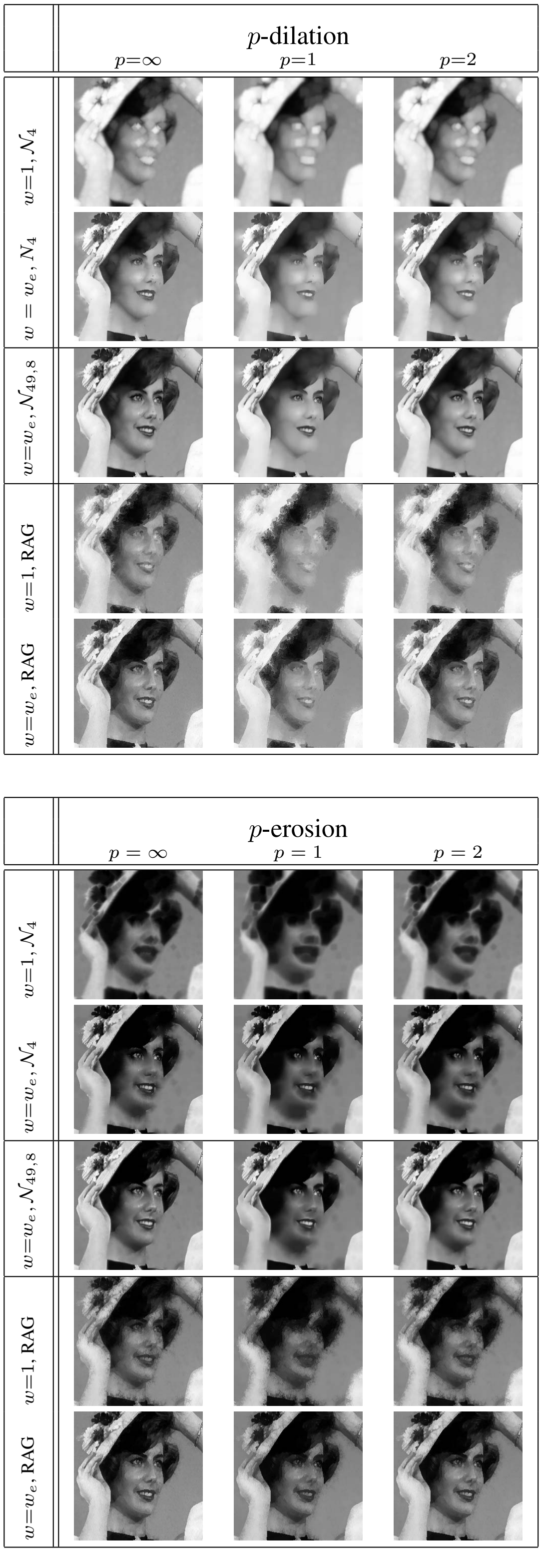



**Fig. 2**. $p$-dilation and $p$-erosion on the image of Fig. 1(a) with different values of $p$, weight functions and graph topologies. See text for more details.

be derived to many other morphological methods such as opening, closing, leveling, etc. Experimental results show the potential of the proposed formalism for the non local processing of images (represented as grid or region adjacency graphs) that better preserves edges and fine structures. Moreover, our formalism can be applied on manifolds and high dimensional discrete data sets that can be represented by a graph. This opens opens new application fields for morphological processing such as machine learning.

## 6. REFERENCES


[1] P. Soille, *Morphological Image Analysis – Principles and Applications*, Springer-Verlag, second edition, 2004.

[2] P. Maragos, "Partial differental equations for morphological scale-spaces and eikonal applications," in *The Image and Video Processing Hanbook*, pp. 587–612. 2005.

[3] M. Breuß, B. Burgeth, and J. Weickert, "Anisotropic continuous-scale morphology," in *IbPRIA*, 2007, vol. LNCS 4478, pp. 515–522.

[4] G. Gilboa and S. Osher, "Nonlocal operators with applications to image processing," CAM report, UCLA, 2007.

[5] A. Elmoataz, O. Lézoray, and S. Bougleux, "Nonlocal discrete regularization an weighted graphs: a framework for image and manifolds processing," HAL TR, Université de Caen Basse-Normandie, 2007.

[6] P. Arbeláez and L. Cohen, "Energy partitions and image segmentation," *JMIV*, vol. 20, no. 1–2, pp. 43–57, 2004.

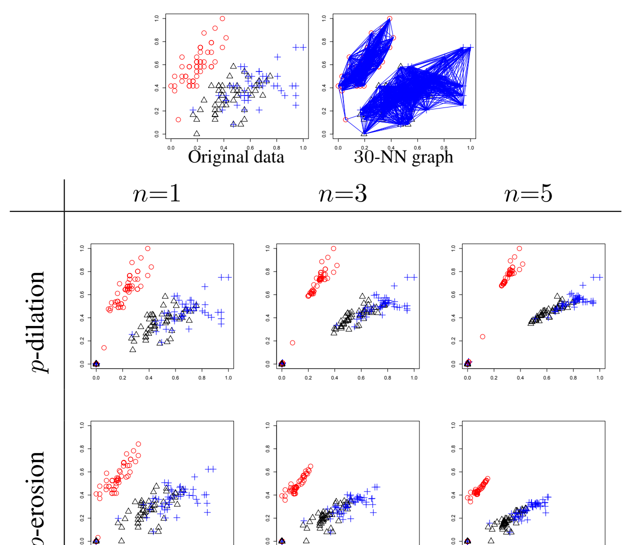




**Fig. 3**. $p$-dilation and $p$-erosion evolution for $p$=1 and different number $n$ of iterations on Iris data set. See text for more details.