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Adaption of the Eikonal Equation over Weighted Graphs

(1) Introduction

The **static eikonal equation** is

$$\begin{cases} \|\nabla f(x)\| - P(x) = 0 & x\in\Omega \subset \mathbb{R}^m \\ f(x) = \phi(x) & x\in\Gamma \subset \Omega \end{cases}, \qquad (1)$$

where ϕ (boundary condition) is a positive speed function defined on Ω, $f(x)$ is the traveling time or distance from source Γ and P a potential function. Solution of (1) represents the shortest distance from x to the zero distance curve given by Γ (where $\phi(x)=0$) and describes the evolution of a front driven by P.

Solutions and numerical schemes for static equation (1) have been proposed: **iterative schemes** [1], **fast sweeping methods** [2] and **fast marching methods** [3,4].

Another approach is to consider a time dependent version:

$$\begin{cases} \partial f(x,t)/\partial t = -\|\nabla f(x)\| + P(x) & x\in\Omega\subset\mathbb{R}^m \\ f(x,t) = \phi(x) & x\in\Gamma\subset\mathbb{R}^m \\ f(x,0) = \phi_0(x) & x\in\Omega \end{cases}. \qquad (2)$$

Contributions: We propose an adaption of (2) over weighted graph of arbitrary structure based on partial difference equations (PdEs).

The analogue of (2) on a weighted graph $G = (V,E,w)$ is

$$\begin{cases} \partial f(u,t)/\partial t = -\|\nabla_w^- f(u)\|_p + P(u) & u\in V \\ f(u,t) = \phi(u) & u\in V_0 \subset V \\ f(u,0) = \phi_0(u) & u\in V \end{cases}, \qquad (3)$$

V is the set of vertices of the graph, V_0 is the initial set of seed vertices, ∇_w^- is a weighted internal morphological gradient on graph and $\|.\|_p$ is the $\mathcal{L}_p$-norm.

Advantages:

- Any domains (high dimensional and irregular) that can be represented by a graph can be considered.
- No spatial discretization or triangulation step.
- For images: local and nonlocal configurations are enabled in a same formulation.
- The formulation recovers well known schemes.

(2) Graphs and Weights

Weighted graph: A *weighted graph* $G=(V,E,w)$ is composed of a set V of *vertices*, a set $E \subset V \times V$ of weighted *edges* and a *weight function* $w : V \times V \to \mathbb{R}^+$. An *edge* uv of E connects two adjacent vertices u and v. The Hilbert space of functions defined on V is noted $\mathcal{H}(V)$.

Neighborhood graphs: Any discrete domain can be represented by a weighted graph where functions of $\mathcal{H}(V)$ represent the data to process. We focus on two neighborhood graphs:

- the k nearest neighbors graphs (k-NNG).
- The τ-neighborhood graph (G_τ).

2D images ($f^0 : V \subset \mathbb{Z}^2 \to \mathbb{R}^m$) can be represented by G_τ graphs. For instance: the 4-**adjacency grid graph** (G_0) with the city block distance. Another useful graph for image is the **region adjacency graph** (RAG) where vertices correspond to image regions and edges regions adjacency relationships.

Graphs weights: **Function w reflects similarities between data**. One can use

$$
\begin{aligned}
g_0(uv) &= 1 \text{ (constant weight case)}, \\
g_1(uv) &= (\rho(F(f^0,u),F(f^0,v)) + \epsilon)^{-1} \text{ or} \\
g_2(uv) &= \exp(-\rho(F(f^0,u),F(f^0,v))^2/\sigma^2)
\end{aligned}
$$

$\sigma > 0$ controls the similarity, ϵ is defined as $\epsilon > 0$, $\epsilon \to 0$, ρ is usually the Euclidean distance.

$F(f^0,u) \in \mathbb{R}^m$ is the features vector of $u \in V$ where $f^0 \in \mathcal{H}(V)$ is an initial function.

Several choices of F can be used. The simplest one is $F(f^0,.) = f^0$. **For images**, another choice is provided by **image patches**: $F(f^0,u) = F_\tau(f^0,u) = \{f^0(v) : v \in V \text{ with } \rho(u,v) \leq \tau\}$ (τ a threshold parameter). This feature vector has been proposed in the context of texture synthesis [5], and further used in the context of image and data processing [6–8].

(3) A Family of Gradients on Weighted Graphs

From [8] that defines for a function $f \in \mathcal{H}(V)$

- the weighted difference operator on graphs for $uv \in E$

$$\partial_v f(u) = \sqrt{w_{uv}}(f(v) - f(u))$$

- the weighted gradient operator for $u \in V$

$$(\nabla_w f)(u) = (\partial_v f(u))_{uv \in E}$$

We define in [9]

- two new weighted directional difference derivatives for an edge $uv \in E$ with $Df(u) = f(v) - f(u)$

$$\textbf{external: } \partial_v^+ f(u) = \sqrt{w_{uv}} \max(0, Df(u))$$
$$\textbf{internal: } \partial_v^- f(u) = \sqrt{w_{uv}} \min(0, Df(u))$$

- two new weighted morphological gradients

$$\textbf{external: } (\nabla_w^+ f)(u) = (\partial_v^+ f(u))_{uv \in E}$$
$$\textbf{internal: } (\nabla_w^- f)(u) = (\partial_v^- f(u))_{uv \in E}$$

The corresponding $\mathcal{L}_p$ $(0<p<\infty)$ and $\mathcal{L}_\infty$ $(p = \infty)$ norms are, respectively

$$\begin{aligned} \|(\nabla_w^\pm f)(u)\|_p &= \Big[\sum_{v \sim u} w_{uv}^{p/2} |(0, Df(u))^\pm|^p\Big]^{1/p} \\ \|(\nabla_w^\pm f)(u)\|_\infty &= \max_{v \sim u}\big(\sqrt{w_{uv}}|(0, Df(u))^\pm|\big) \end{aligned} \tag{4}$$

where $v \sim u$ means that v is neighbor of u. These gradients norms have the following properties

$$\begin{aligned} \|(\nabla_w f)(u)\|_p^p &= \|(\nabla_w^+ f)(u)\|_p^p + \|(\nabla_w^- f)(u)\|_p^p \\ \|(\nabla_w f)(u)\|_\infty &= \max(\|(\nabla_w^+ f)(u)\|_\infty, \|(\nabla_w^- f)(u)\|_\infty) \end{aligned}$$

Properties:

- These general definitions are defined on graphs of arbitrary topology.
- They can be used to process any discrete regular or irregular data sets that can be represented by a weighted graph.
- Local and nonlocal settings are directly handled in these definitions and both are expressed by the graph topology in terms of neighborhood connectivity [10].

(4) Numerical Schemes for Morphological Processes

Time dependent eikonal equation (2) is linked with mathematical morphology processes and can be viewed as morphological evolution equations.

Dilation and Erosion: The two fundamental operations in mathematical morphology are **dilations** $\delta:\mathbb{R}^m \to \mathbb{R}^m$ and **erosions** $\varepsilon:\mathbb{R}^m \to \mathbb{R}^m$. Classically, these operations are performed by considering lattices and their implementations are **algebraic (discrete)**.

Continuous morphology [11] defines **flat** dilation and erosion of a function $f:\mathbb{R}^m \to \mathbb{R}$ with the following partial differential equations (PDEs):

$$\partial_t \delta(f) = \partial_t f = +\|\nabla f\|_p \quad \text{and} \quad \partial_t \varepsilon(f) = \partial_t f = -\|\nabla f\|_p$$

Analogues on Weighted Graphs: In [9] we have proposed analogues over graphs of the continuous morphological equations by using morphological gradients and their numerical schemes (4). For a graph $G=(V,E,w)$, a function $f \in \mathcal{H}(V)$

$$\begin{aligned} \partial_t \delta(f(u)) &= \partial_t f(u) = +\|(\nabla_w^+ f)(u)\|_p \\ \partial_t \varepsilon(f(u)) &= \partial_t f(u) = -\|(\nabla_w^- f)(u)\|_p \end{aligned} \tag{5}$$

These equations constitute a morphological framework [9] based on PdEs that extends algebraic and continuous morphological operators for images and high dimensional data processing.

Relations with Algebraic Formulations: For the particular case where $w = g_0 = 1$ and with the $\mathcal{L}_\infty$-norm, our definitions recover the definitions of algebraic:

- classical internal and external differences and gradients definitions.
- morphological gradient operator $\|(\nabla_w^+ f)(u)\|_\infty + \|(\nabla_w^- f)(u)\|_\infty$.
- morphological Laplace operator $\|(\nabla_w^+ f)(u)\|_\infty - \|(\nabla_w^- f)(u)\|_\infty$.
- dilation and erosion on graphs.

(5) Eikonal Equation over Weighted Graphs

System (2) can be viewed as an erosion process regarding the minus sign and a constant potential function P. The adaption of the eikonal equation on graphs can be directly obtained with the erosion process defined in (5) and by replacing operator ∇f by the internal gradient $\nabla^-_w f$. The analogue of (2) over graphs (2) (parameterized by p and w) is

$$\begin{cases} \partial f(u,t)/\partial t = -\|(\nabla^-_w f)(u)\|_p + P(u) & u \in V \\ f(u,t) = \phi(u) & u \in V_0 \subset V \\ f(u,0) = \phi_0(u) & u \in V \end{cases} \tag{6}$$

where V_0 corresponds to the initial set of seed vertices.

Numerical Schemes and Algorithms: With our definitions of morphological gradients, we can directly obtain the numerical schemes to solve (6) without any spatial discretization thanks to the discrete form of our operators on graphs. With $f^n(u) \approx f(u, n\Delta t)$, we have

$$\frac{f^{n+1}(u) - f^n(u)}{\Delta t} = -\|(\nabla^-_w f^n)(u)\|_p + P(u). \tag{7}$$

With the corresponding $\mathcal{L}_p$ ($0<p<\infty$) and $\mathcal{L}_\infty$ ($p = \infty$) norms, we have

$$\frac{f^{n+1}(u) - f^n(u)}{\Delta t} = -\left[\sum_{v\sim u} w_{uv}^{p/2} |\min(0, Df^n(u))|^p\right]^{\frac{1}{p}} + P(u) \tag{8}$$

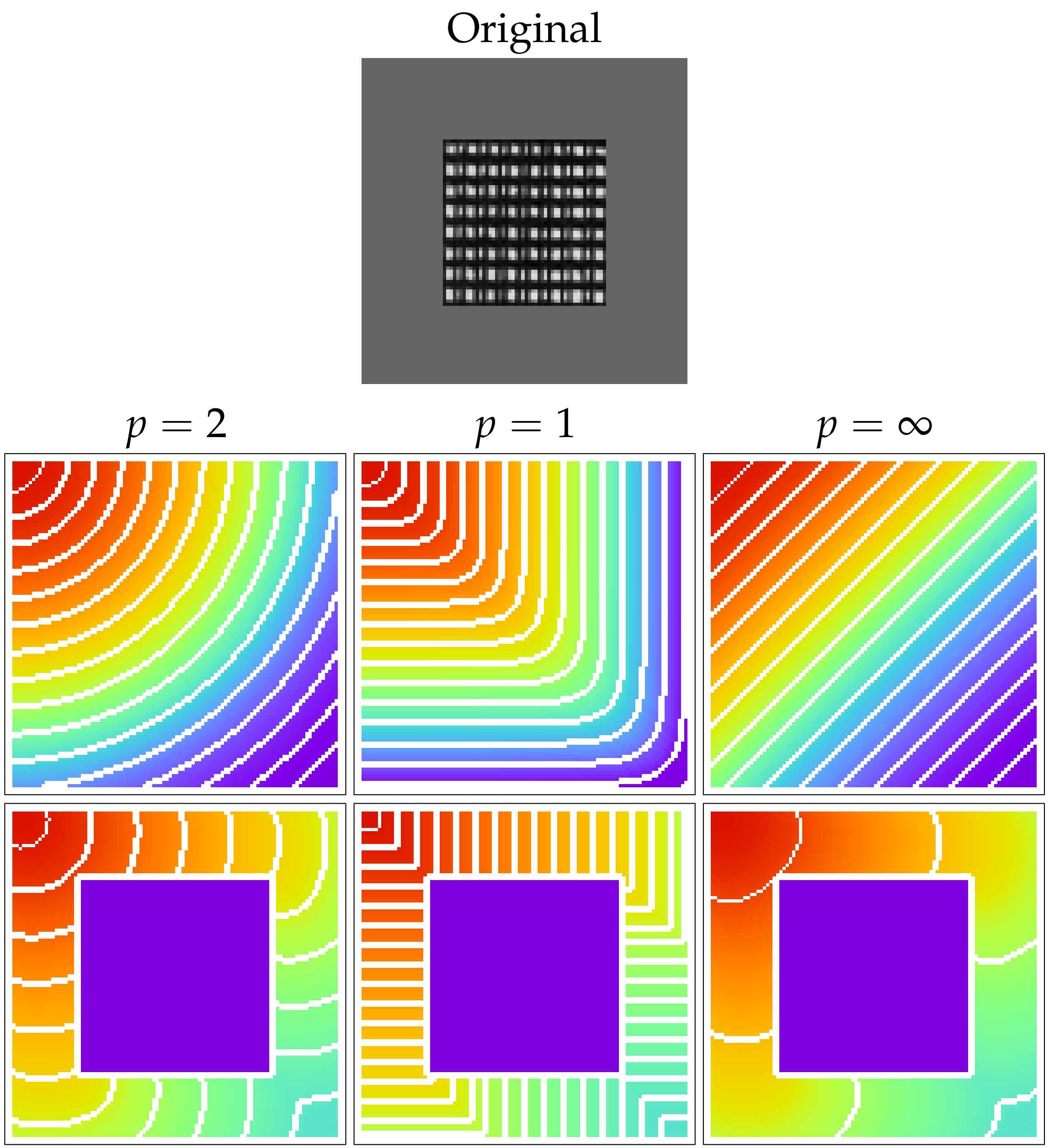
$$\frac{f^{n+1}(u) - f^n(u)}{\Delta t} = -\max_{v\sim u}(\sqrt{w_{uv}} |\min(0, Df^n(u))|) + P(u) \tag{9}$$

These numerical schemes work on any graph of arbitrary topology. This implies that our formulation constitutes a simple and unified method to solve the eikonal equation for any data defined on regular or irregular domains that can be modeled by a graph in any dimensions.

Relations with other Schemes: With specific parameters and graphs, the proposed formulation recovers the **Osher-Sethian discretization scheme** for a m-dimensional grid (with (8) when $p = 2$) and a **Dijkstra like algorithm** (with (9) when $p = \infty$) for any graphs methods.

(6a) Experiments — Distance Computation

Images



Adaptive Framework for Generalized Weighted Distance Computation with different p values (distance maps with iso-levels superimposed).

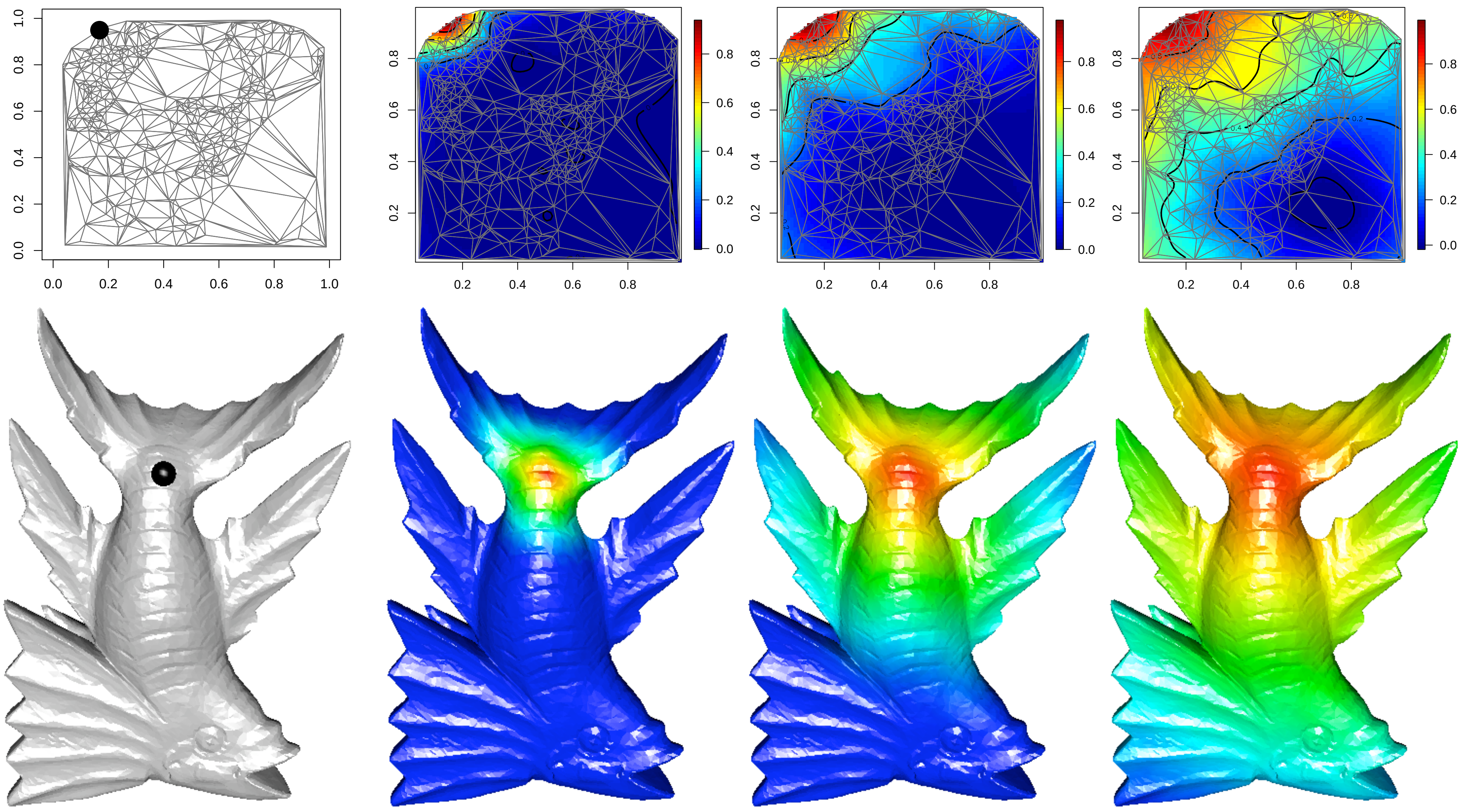
Potential $P = 1$ and graph configurations are:

- first row (local): G_0, $w{=}g_0$, $F{=}f^0$
- second row (nonlocal): G_2, $w{=}g_2$, $F{=}F_5(f^0,.)$

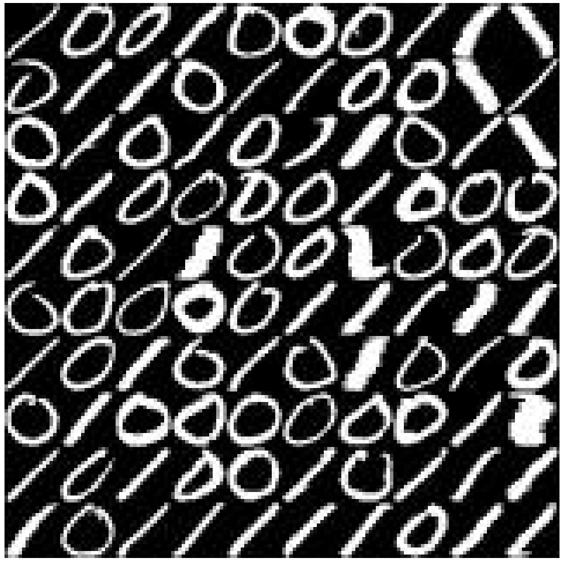
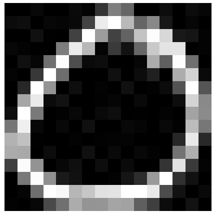
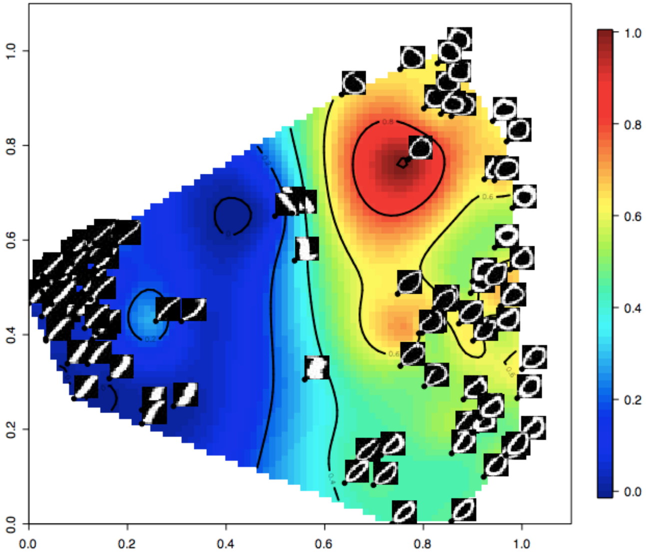
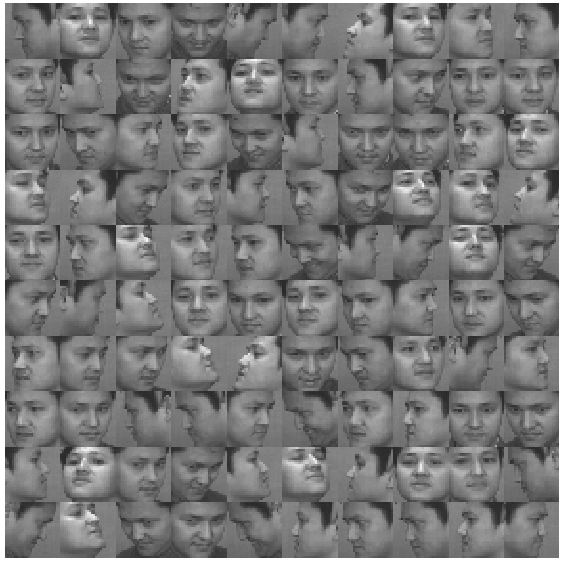
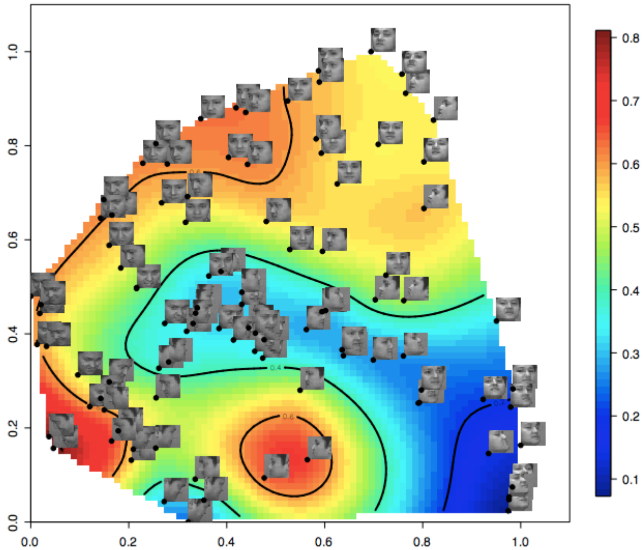
(6b) Experiments — Distance Computation

Delaunay Graphs and Meshes

Original graph	Left to right: front evolution



Unorganized High Dimensional Data: Data Ranking (1)

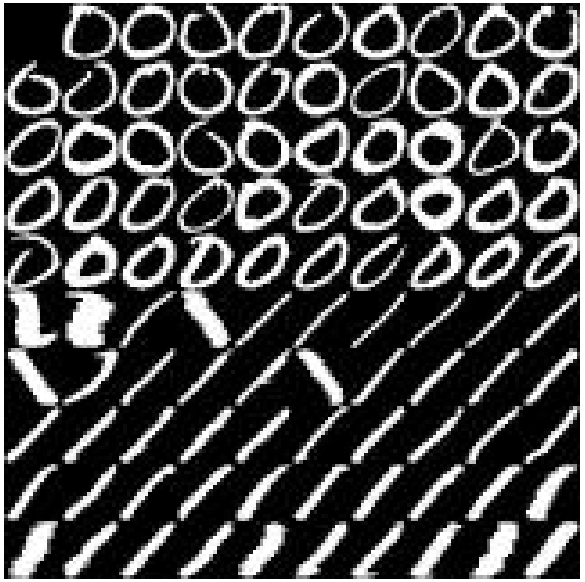
Original data sets	Initial Seed	Distance Map
		
		

Graphs configurations ($|V| = 100$):

- first row: 20-NNG, $w = g_1$, $f^0 : V \to \mathbb{R}^{16\times16}$
- second row: 3-NNG, $w = g_1$, $f^0 : V \to \mathbb{R}^{29\times29}$

(6c) Experiments — Distance Computation

Unorganized High Dimensional Data: Data Ranking (2)

Ordered Ranking Results	10 Closest (at top) and 10 Farthest (at bottom) Results
	
	

(6d) Experiments — Nonlocal Segmentation: Images

Original+seeds	Potential P	Local	Nonlocal+patches

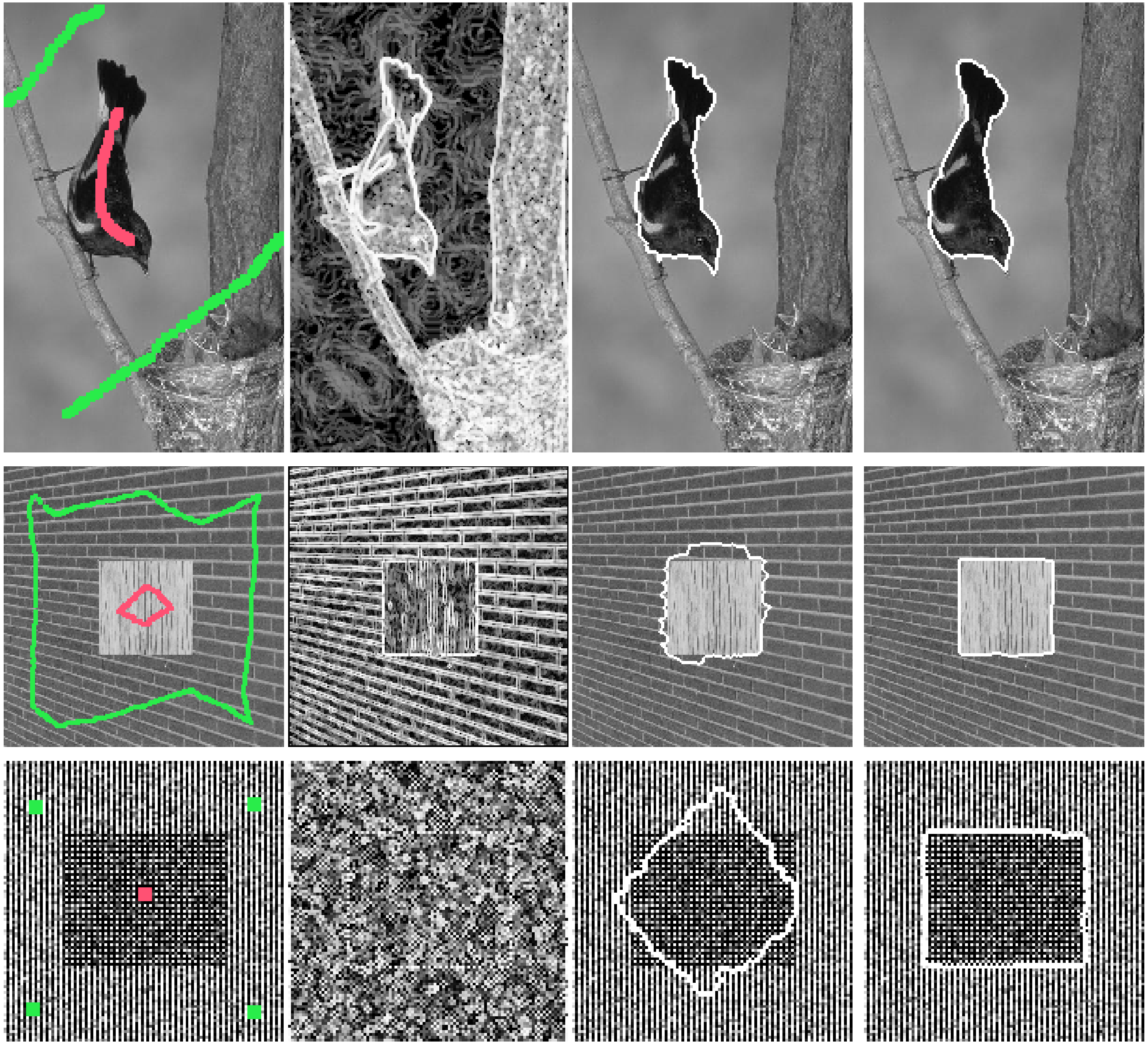


Image segmentation with class membership computation. Potential P is original image gradient.

Graphs configurations:

local case: G_0, $w = g_0$, $F = f^0$,

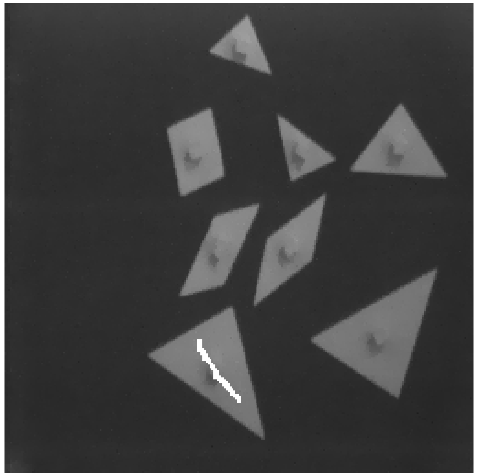
nonlocal case:

- First and second rows: G_3, $w = g_2$, $F = F_2(f^0, .)$.
- Third: $G_0 \cup$ 4-NNG, $w = g_1$, $F = F_3(f^0, .)$.

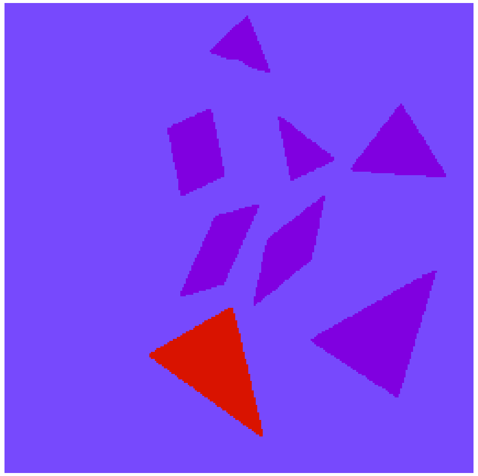
(6e) Experiments — Nonlocal Segmentation

Region Based Graphs (1)

Original Image



Distance Map



Local segmentation

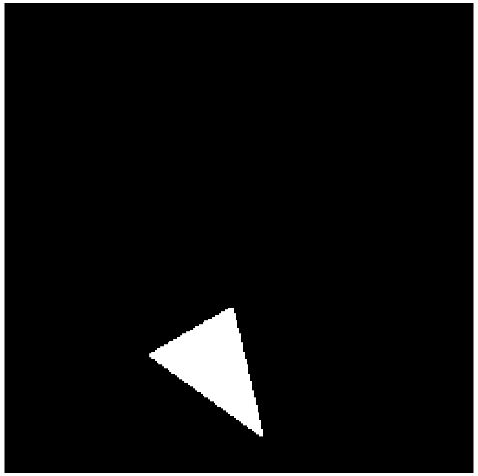
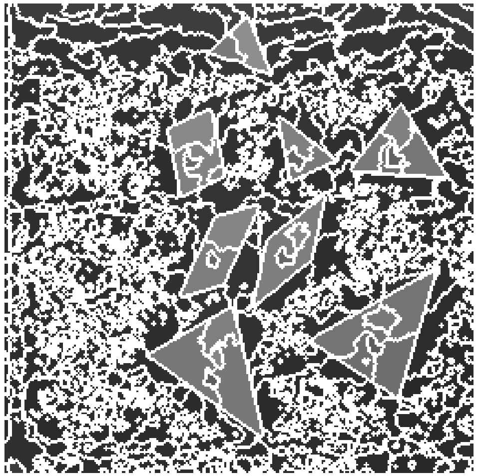
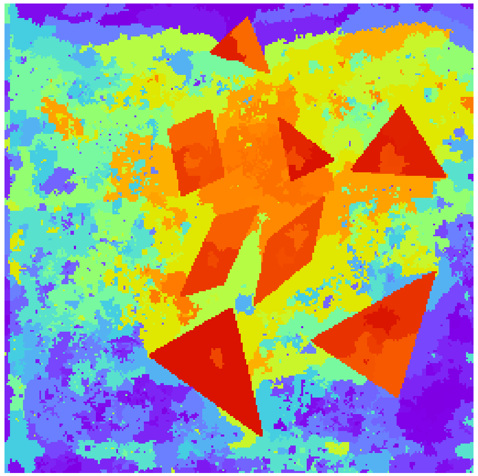


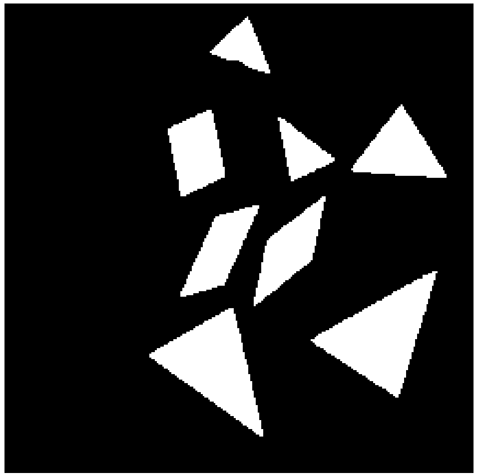
Image Partition



Distance Map



Nonlocal segmentation



Segmentation by distance map thresholding.

Advantages of regions based graph:

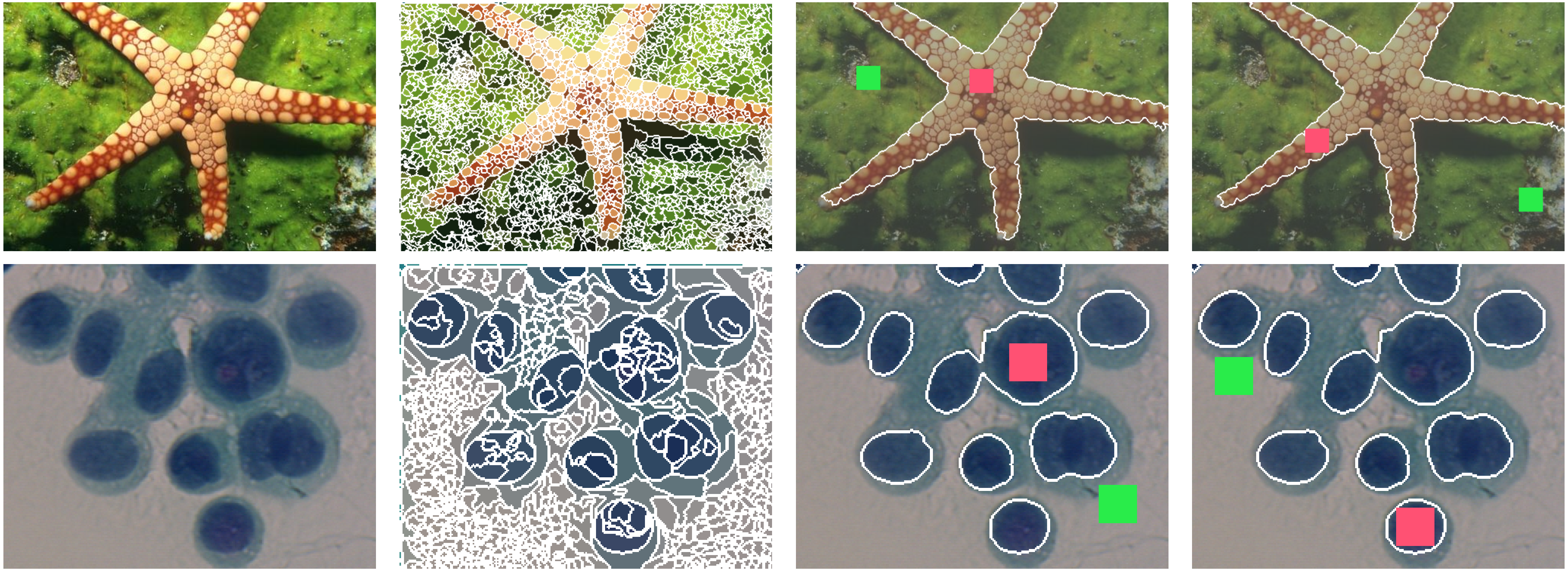
- Nonlocal object segmentation.
- Minimal number of seeds.
- Fast computation: original image, $|V| = 65\,536$ pixels; image partition, $|V| = 1\,324$ regions 98% of reduction in terms of vertices.

Graph configurations:

- local case: RAG, $w = g_2$
- nonlocal case: RAG $\cup$ 5 - NNG, $w = g_2$

(6e) Experiments — Nonlocal Segmentation
Region Based Graphs (2)

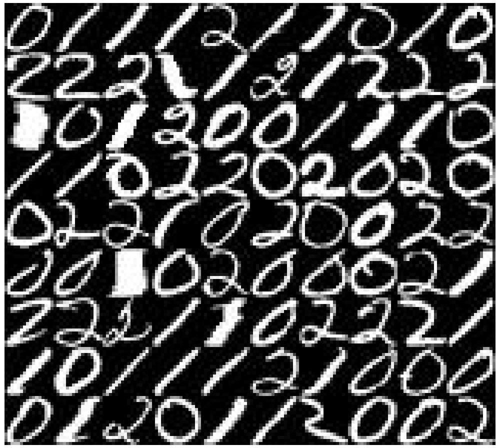
Original Image	Image Partition	Segmentation + initial seeds



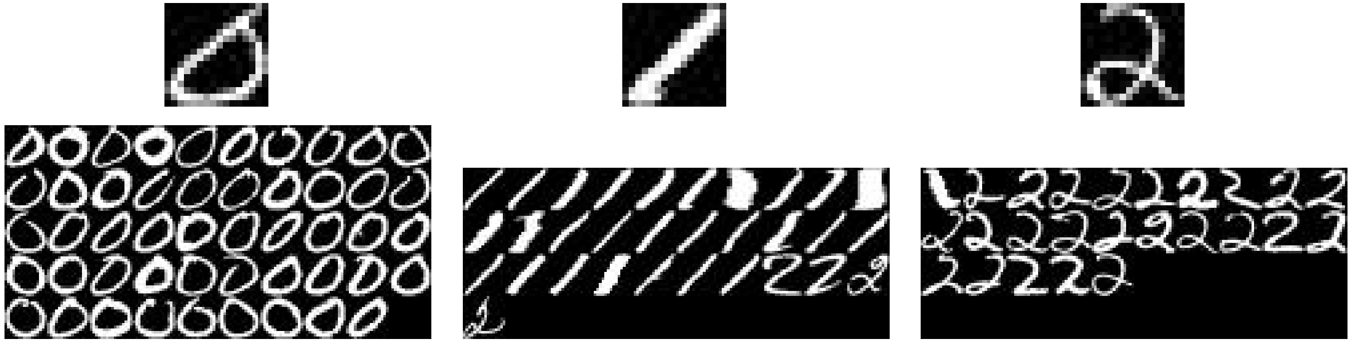
Graphs configuration: RAG $\cup$ 4-NNG, $w = g_2$.

(6f) Experiments — Nonlocal Segmentation: Unorganized High Dimensional Data Clustering and Classification

Original set

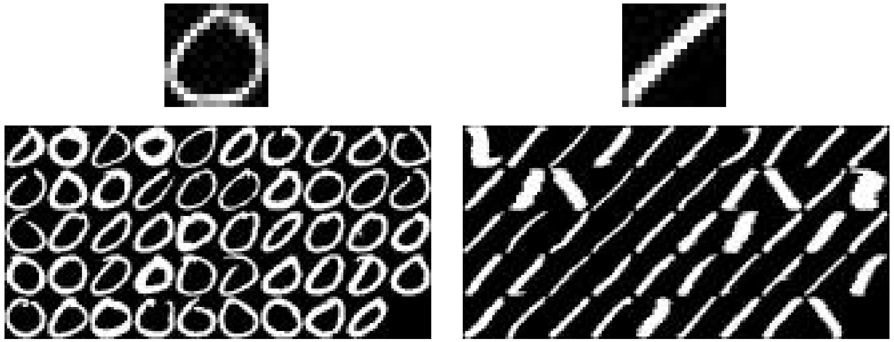


Seeds and Classification Results



Original set

Seeds and Classification Results



Graphs configuration: first row 20-NNG, second row: 5-NNG, for both $w = g_1$ and $f^0 : V \to \mathbb{R}^{16}$

Classification rates: 93.1% and 100%

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