NONLOCAL MORPHOLOGICAL LEVELINGS BY PARTIAL DIFFERENCE EQUATIONS OVER WEIGHTED GRAPHS

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1 Introduction

The two fundamental operations in Mathematical Morphology (MM) are **dilations** and **erosions**. They employ so-called *structuring elements*. Classically, such operators are performed by considering sets and lattices operations. Their implementations are usually relying on algebraic (discrete) settings.

Continuous morphology

Alternative formulations [1] based on nonlinear *partial difference equations* (PDEs) have been proposed. They show that multiscale dilations δ and erosions ε of a function $f^0:\mathbb{R}^n \to \mathbb{R}$ by a structuring element $B=\{x\in\mathbb{R}^n : \|x\|_p \leq 1\}$ can be generated by these PDEs

$$\delta(f) = \partial_t f = +\|\nabla f\|_p \quad \text{and} \quad \varepsilon(f) = \partial_t f = -\|\nabla f\|_p \ ,$$

with $f=f^0$ at $t=0$, ∇ is the gradient operator and $\|.\|_p$ is the $\mathcal{L}_p$-norm.

Motivations

The PDEs framework has **advantages** (better geometry approximation and subpixel accuracy) and **drawbacks** (numerical discretization is difficult for high or irregular domains and only local derivatives are considered). Moreover, MM provides well known methods for images but **there no exist general extension** for the processing of high dimensional unorganized data.

Contributions

PDEs based operations are **extended** to *nonlocal discrete schemes* by **partial difference equations (PdEs)** overs graphs.

The proposed formulation has advantages. **Local and nonlocal processing** are naturally and directly enabled within a same formulation. **Any discrete domain** that can be described by a graph can be considered. These points provide **new insights** of morphological operations to unorganized data and to nonlocal image processing.

2 The ingredients

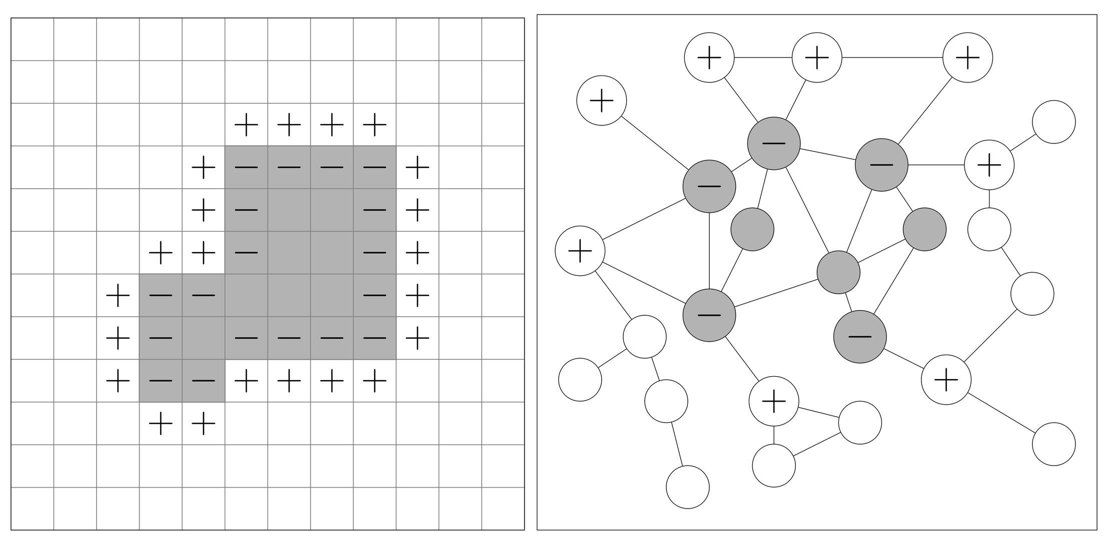
Weighted graph

A *weighted graph* $G=(V,E,w)$ is composed of a set V of *vertices*, a set $E\subset V\times V$ of weighted *edges* and a *weight function* $w:V\times V\to\mathbb{R}^+$. An *edge* uv of E connects two adjacent vertices u and v. The Hilbert space of functions defined on V is noted $\mathcal{H}(V)$.

Graph boundary sets

Let $\mathcal{A}$ (gray vertices) be a set of connected vertices with $\mathcal{A}\subset V$ ($\mathcal{A}^c$ is its complement) such that for all $u\in\mathcal{A}$, there exists a vertex $v\in\mathcal{A}$ with $uv\in E$. Then we define

the outer set: $\partial^+\mathcal{A} = \{u\in\mathcal{A}^c : \exists v\in\mathcal{A} \text{ with } uv\in E\}$ and
the inner set: $\partial^-\mathcal{A} = \{u\in\mathcal{A} : \exists v\in\mathcal{A}^c \text{ with } uv\in E\}$.



(a) 4-adjacency grid graph (b) Arbitrary graph

Discrete gradients on graphs and norms

From [2,3] works, we define **two weighted directional discrete gradients** of a function $f\in\mathcal{H}(V)$ at a vertex $u\in V$ by

$$(\nabla_w^+ f)(u) = (\partial_v^+ f(u))_{uv\in E} = (\max(0, w_{uv}^{1/2}(f(v)-f(u))))_{uv\in E}$$
$$(\nabla_w^- f)(u) = (\partial_v^- f(u))_{uv\in E} = (\min(0, w_{uv}^{1/2}(f(v)-f(u))))_{uv\in E}$$

The $\mathcal{L}_p$-norm and $\mathcal{L}_\infty$-norm of $(\nabla_w^+ f)(u)$ are

$$\|(\nabla_w^+ f)(u)\|_p = \left[\sum_{v\sim u} w_{uv}^{p/2} \left|\max\big(0, f(v)-f(u)\big)\right|^p\right]^{1/p}$$

$$\|(\nabla_w^+ f)(u)\|_\infty = \max_{v\sim u}\left(w_{uv}^{1/2}\left|\max\big(0, f(v)-f(u)\big)\right|\right).$$

Norms of $(\nabla_w^- f)(u)$ is obtained by replacing the max operator by the min operator.

3 From continuous to discrete

With the **two discrete gradients** and the **graph boundary sets**, dilation and erosion can be interpreted as

- A growth process that *adds* vertices and *maximize* a surface gain proportional to $\|\nabla_w^+\|_p$ **(dilation)**.
- A contraction process that *removes* vertices and *minimize* a surface gain proportional $\|\nabla_w^-\|_p$ **(erosion)**.

Dilation and erosion reformulation

Then, the **reformulation** of *continuous* dilations δ and erosions ε over **any weighted graph** is

$$\delta(f(u)) = \partial_t f(u) = +\|(\nabla_w^+ f)(u)\|_p \text{ and}$$
$$\varepsilon(f(u)) = \partial_t f(u) = -\|(\nabla_w^- f)(u)\|_p \ .$$

On the contrary to PDEs case, **no spatial discretization is needed** thanks to derivatives directly expressed in a discrete form. With $f^n(u)\approx f(u, n\Delta t)$, $\forall u\in V$ and at time $n+1$, we have the iterative algorithms

$$\text{dilations: } f^{n+1}(u) = f^n(u) + \Delta t \|(\nabla_w^+ f)^n(u)\|_p$$
$$\text{erosions: } f^{n+1}(u) = f^n(u) - \Delta t \|(\nabla_w^- f)^n(u)\|_p$$

with $f^{n=0}=f^0$ and where $f^0\in\mathcal{H}(V)$ is the initial function defined on the graph vertices.

Relation with other methods

With specific parameters and graphs, the proposed formulation recovers (1) the **Osher-Sethian discretization scheme** used in continuous morphology implementation and (2) the **classical algebraic flat morphological dilation and erosion**.

Levelings reformulation

[4] introduced nonlinear PDEs that model levelings. The **discrete analogue** of such continuous formulation over graphs is, $\forall u\in V$

$$\partial_t f(u) = \mathrm{sgn}(f^0(u) - f(u)) \|(\nabla_w^\pm f)(u)\|_p$$

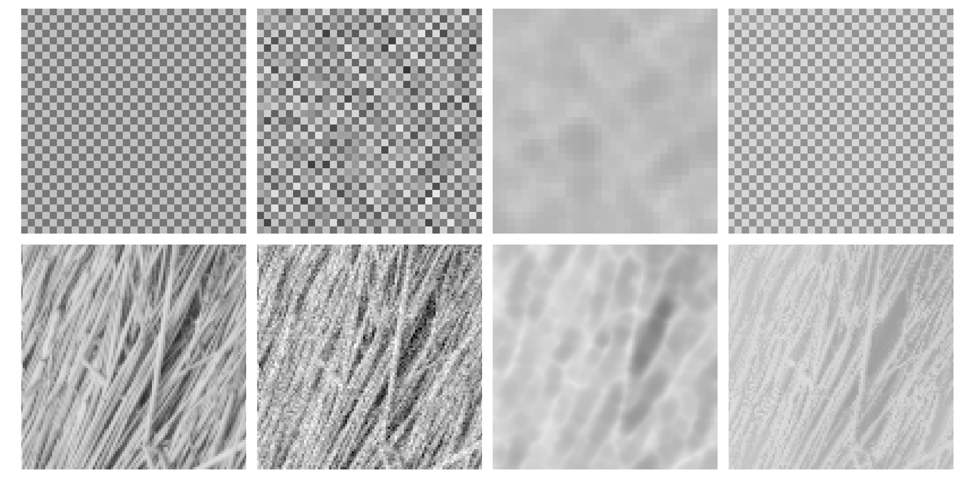
with $f=m$ at $t=0$. $f^0\in\mathcal{H}(V)$ is a *reference function* and $m\in\mathcal{H}(V)$ is a *marker function* from which levelings can be produced. The sign function sgn controls the evolution:

- If $f^0 > f$ then the process acts as **dilation** (with $\nabla_w^\pm = \nabla_w^+$).
- If $f^0 < f$ then the process acts as **erosion** (with $\nabla_w^\pm = \nabla_w^-$).

4 Adaptive morphology in image processing

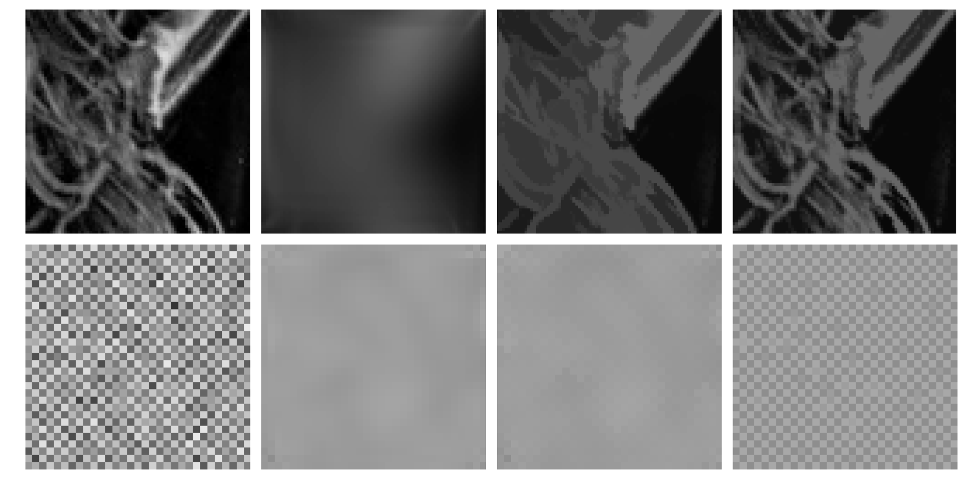
Nonlocal with patches vs. local

Closing operation (dilations then erosions)



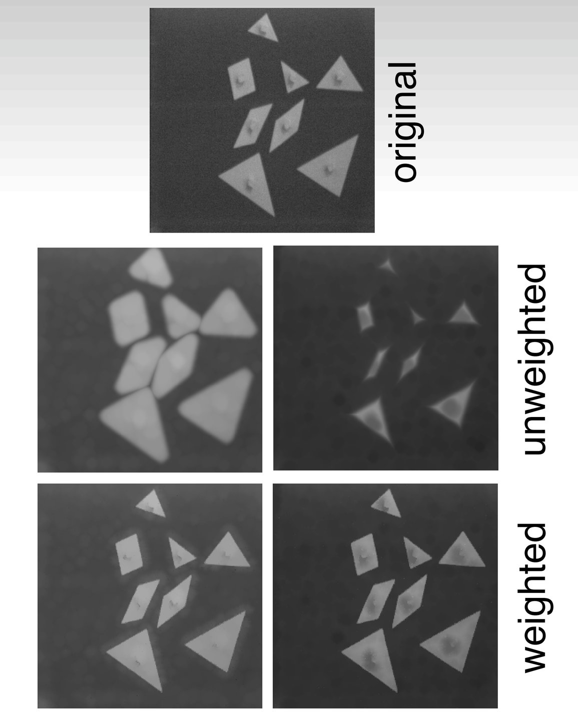
original noisy local nonlocal

Levelings



original marker local nonlocal

Weighted vs. Unweighted



dilations erosions

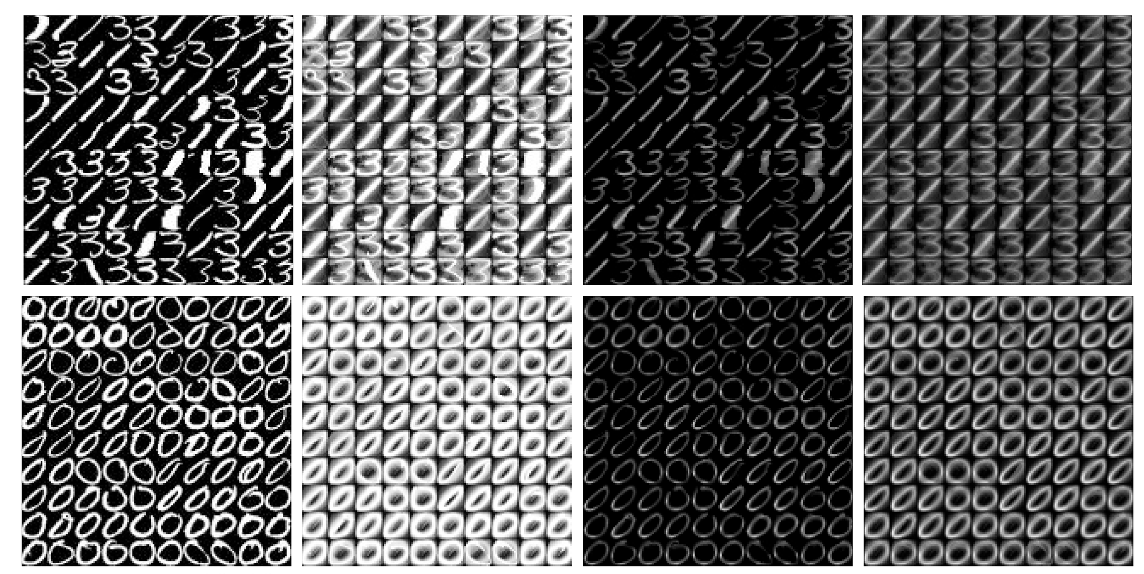
5 Extension to arbitrary graphs

Fast image processing with Region Adjacency Graph (RAG)

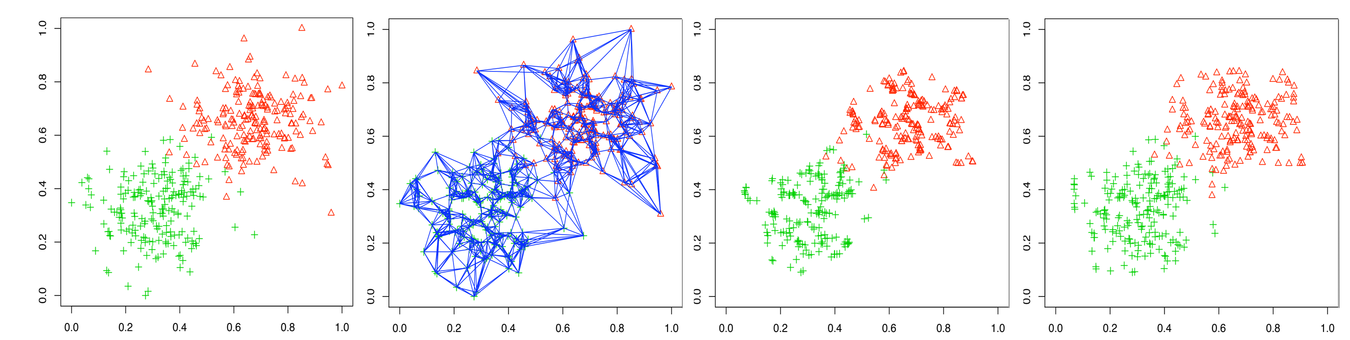
Fast multiscale RAG Levelings

original | partition (87% of reduction) | mean color | marker 1 | levelings 1 | marker 2 | levelings 2 | partition (82% of reduction) | mean color | dilations | erosions

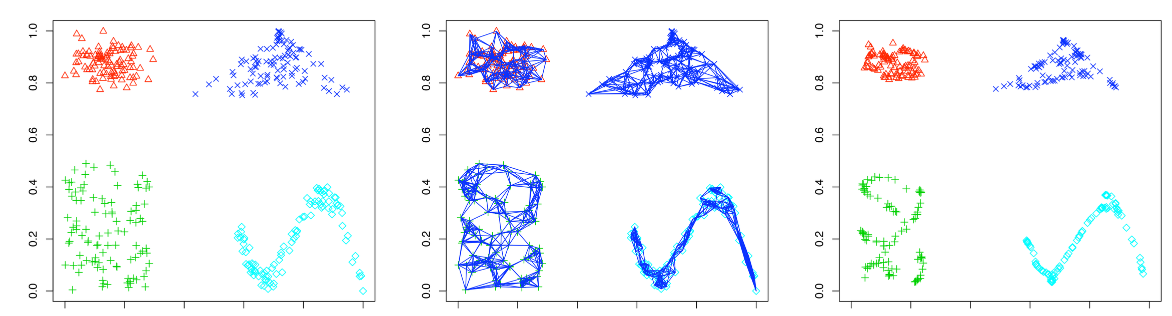
Morphological processing of unorganized data



original ($\mathbb{R}^{16\times16}$) dilations erosions closings



original 8-NN graph marker levelings



original (4 data sets) 8-NN graphs closings

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[2] S. Bougleux, A. Elmoataz, and M. Melkemi, "Discrete regularization on weighted graphs for image and mesh filtering," in *SSVM*, ser. LNCS, vol. 4485, 2007, pp. 128–139.

[3] A. Elmoataz, O. Lézoray, and S. Bougleux, "Nonlocal discrete regularization an weighted graphs: a framework for image and manifolds processing," *IEEE TIP*, vol. 17, no. 7, pp. 1047–1060, 2008.

[4] P. Maragos and F. Meyer, "Nonlinear PDEs and numerical algorithms for modeling levelings and reconstruction filters," in *Scale-Space*, ser. LNCS, vol. 1682, 1999, pp. 363–374.

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